Technical Report No. 32-650 (Part I)

The Rotating Superconductor
Part I: The Fluxoid

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CONTENTS

ABSTRACT

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The magnetic field present in the bore of a hollow, rotating superconducting cylinder is deduced from London's theory of superconductivity. For a cylinder whose transverse dimensions are larger than a penetration depth, the results show that the quantity $\omega_L + \omega$ is a constant. Here, ω_L is the angular Larmor frequency of the electron, a measure of the magnetic field in the bore (in fact, proportional to this field); ω is the angular velocity of the cylinder itself. The constant is the value of $\omega_L + \omega$ at the time the superconductor becomes superconducting. A nucleation model of the superconducting transition is used to deduce this relation. The Meissner effect in rotating superconductors is discussed, and it is pointed out how the effects of an applied field on a superconductor tend to be cancelled by its rotation. It is pointed out that this cancellation may be important for observing superconductivity in superconductors having tiny critical fields.

Author

I. INTRODUCTION

Recently, Hildebrandt (Ref. 1) has investigated experimentally the magnetic fields associated with a rotating, hollow, superconducting cylinder. As is shown in this Report, the values of the field he observes can be related to the fluxoid (Ref. 2) associated with such a cylinder. The point then becomes to show how the requisite values of the fluxoid are generated. This is done by means of an argument in which the actual dynamics of the superconducting transition are considered. A thermodynamic argument then shows that the resulting value of the fluxoid is the one that yields the most stable state, thermodynamically, for the superconductor. The expression for the free energy that is used in this argument is derived and discussed in Part II of this series of reports (Ref. 3).

This Report presents the exact solutions of London's equations for a rotating hollow cylinder of infinite length, with special attention to the dependence of the solution on the fluxoid. The results are then applied to an arbitrary hollow cylinder whose inner diameter and thickness are both large compared to the penetration depth, these being the conditions satisfied by the cylinders in Hildebrandt's experiments. (The cylinder is still considered to be infinite in length, however.)

The principal result is that the magnetic field B in the hollow of the cylinder is given by $|e|B/2mc = -(\omega - \omega_0')$. Here, $|e|B/2mc$ is the Larmor frequency of the electron in rad/sec, ω is the angular velocity of the cylinder, and $\omega_0' = \omega_0 + e/2mcB_\infty$ is a quantity that is determined at the time the cylinder becomes superconducting, where ω_0 is the angular velocity of the cylinder at the time of the transition and B_∞ is the applied field then present.

In addition, we also derive from our solution the proportionality that holds between fluxoid and flux and evaluate it in the limit of a thin-walled cylinder. The proportionality is found to agree with expressions derived by other means. Also mentioned in the Report is the Meissner effect in rotating superconductors and the application of this effect to the observation of superconductivity in superconductors having tiny critical fields.

II. LONDON'S EQUATION IN A ROTATING SUPERCONDUCTOR AND THE MEISSNER EFFECT

Our procedure is first to write the solution for London's equations as they apply to the rotating hollow cylinder and then to express this solution in terms of the fluxoid and the applied field. Arguments are then given that indicate the value to be assigned to the fluxoid. With a value assigned the fluxoid, the solution of London's equation becomes fully determined.

London's equation is

$$\nabla \times \mathbf{v}_s(\mathbf{r}) = -\frac{e}{mc}\mathbf{B}$$

where $\mathbf{v}_s(\mathbf{r})$ is the superelectron velocity field, $\mathbf{B}$ the magnetic field, and e/m refers to the electron. Since $\mathbf{B}$ always satisfies $\nabla \times \mathbf{B} = (4\pi/c)\,\mathbf{j}$, then, in the superconductor,

$$\lambda^2 \nabla \times \nabla \times \mathbf{B} + \frac{mc}{e}\nabla \times \mathbf{v}_l(\mathbf{r}) + \mathbf{B} = 0 \qquad \text{(1a)}$$

where

$$\lambda^2 = \frac{mc^2}{4\pi e^2 \rho_0}$$

ρ_0 = superelectron density

$\mathbf{v}_l(\mathbf{r})$ = velocity field of the lattice (lattice = superconductor minus superelectrons)

Outside the superconductor,

$$\nabla \times \nabla \times \mathbf{B} = 0 \qquad \text{(1b)}$$

with $\mathbf{B}$ continuous across the surface of the superconductor.

For a superconductor rotating with a constant angular velocity, $\boldsymbol{\omega}$, $\mathbf{v}_l(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$, these equations can be written in the form

$$\lambda^2 \nabla \times \nabla \times (\mathbf{B} + \mathbf{B}_0) + (\mathbf{B} + \mathbf{B}_0) = 0 \quad \mathbf{B}_0 = \frac{2mc}{e}\boldsymbol{\omega} \qquad \text{(2)}$$

III. SOLUTIONS FOR THE INFINITELY LONG HOLLOW CYLINDER

The more familiar London equation for a stationary superconductor is immediately recovered by setting $\omega = 0$, and we see that to every solution $\mathbf{B} = \mathbf{f}(\mathbf{r})$ for the stationary superconductor, there is a corresponding solution $\mathbf{B} = \mathbf{f}(\mathbf{r}) - \mathbf{B}_0$ for the rotating superconductor. Thus, for example, the solution $\mathbf{B} = -\mathbf{B}_0$ everywhere for the rotating superconductor corresponds to the solution $\mathbf{B} = 0$ everywhere for the stationary superconductor. What is striking is that whereas in the stationary superconductor this field $\mathbf{B}_0$ would penetrate only to a distance λ into the superconductor, in a rotating superconductor it permeates the superconductor, its value undiminished from the value it had outside. We see therefore that the Meissner effect in a rotating superconductor must be quite different from that in a stationary superconductor. The Meissner effect in a stationary superconductor requires that even if the field outside the superconductor is nonzero, the field inside the superconductor must remain zero, except within a penetration depth λ of the surface. This is, of course, the familiar exclusion of flux from the interior of a bulk superconductor. However, by the correspondence of solutions that we have shown to exist between the rotating and the stationary superconductor, we see that the Meissner effect in a rotating superconductor now requires that even though the value of the external field is changed from $-\mathbf{B}_0$, the interior field still remains $-\mathbf{B}_0$ except within a penetration depth λ. Both phenomena can be described as a Meissner effect. We see now, however, that the Meissner effect is not so much an exclusion of flux from the bulk of the superconductor as it is an exclusion of net current.*

*The solution $\mathbf{B} = -\mathbf{B}_0$, besides serving to contrast the Meissner effect as it appears in stationary and in rotating superconductors, may also indicate an important use of rotation of superconductors. For this solution, both the field configuration and the electron motion for the rotating superconductor and for the rotating normal conductor are identical, so that the free energy difference between normal and superconducting states does not depend on the external field. This means that even if a stationary superconductor is placed in a uniform field $\mathbf{B}$ that is greater than the critical field, rotation of the specimen at an angular velocity $\boldsymbol{\omega}$ ($\boldsymbol{\omega} = -(2mc/e)\,\mathbf{B}$) should allow it to become superconducting provided that the temperature of the superconductor is below the critical temperature. In a sense, then, rotating the superconductor "bucks out" the magnetic field.

This "bucking out" may be useful in establishing the superconductivity of materials with tiny critical fields. After ambient fields have been reduced as far as is possible by conventional means, the effects of tiny residual fields can ultimately be cancelled by rotation of the specimen.

We now go on to the solutions of London's equation for the cylinder. In cylindrical coordinates, with ρ the radial coordinate and with $\mathbf{B} + \mathbf{B}_0$ along the z-axis, Eq. (2) becomes

$$-\frac{\lambda^2}{\rho}\frac{\partial}{\partial \rho}\left[\rho\,\frac{\partial (B + B_0)}{\partial \rho}\right] + (B + B_0) = 0 \tag{3}$$

We now change to a dimensionless variable $r = \rho/\lambda$, write $b(r) = B(r) + B_0$, and obtain

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{db}{dr}\right) + b = 0 \tag{4}$$

This equation has as its solutions the hyperbolic Bessel functions of zero order:

$$b(r) = \alpha I_0(r) + \beta K_0(r) \tag{5}$$

where

$$I_n(r) = J_n(ir)$$
$$K_n(r) = H_n^{(1)}(ir)$$

The function $K_0(r)$, which behaves as $\ln r$ + const for $r = 0$, must be excluded for the case of a solid cylinder. The solution for the solid cylinder of radius R is thus

$$B + B_0 = \left[B(R) + B_0\right]\frac{I_0\left(\dfrac{r}{\lambda}\right)}{I_0\left(\dfrac{R}{\lambda}\right)} \tag{6}$$

If $R, \rho \gg \lambda$, then,

$$B + B_0 \cong \left[B(R) + B_0\right]\frac{R}{\rho}\exp\left[-\frac{1}{\lambda}(R - \rho)\right] \tag{7}$$

so that for $\rho \ll R$ in the interior of the cylinder, regardless of $B(R)$ (the value of the applied axial field),

$$B(\rho) + B_0 = 0$$

We see from this solution what we anticipated before: just as in the stationary superconductor surface currents arise that tend to screen any external field (less than critical, of course) from the interior where the field remains zero, so in the rotating superconductor, the surface currents also screen the external field but leave the interior with the field $-B_0$.

For the hollow cylinder, however, both Bessel functions must be retained. The boundary condition to be imposed on the interior wall of the cylinder at $R_i\lambda$ is most conveniently expressed in terms of the fluxoid. The fluxoid, defined as

$$\oint \left[m\,\mathbf{v}_s(\mathbf{r}) + \frac{e}{c}\mathbf{A}(\mathbf{r}) \right] \cdot d\boldsymbol{\lambda} \tag{8}$$

is a constant that is fixed at the moment the superconductor becomes superconducting; and it remains constant so long as the superconductor is in a stationary state (Ref. 2). For the cylinder,

$$\frac{e}{c} \oint_{|\mathbf{r}|=\lambda R_i} \mathbf{A}(\mathbf{r}) \cdot d\boldsymbol{\lambda} = \frac{e}{c}\pi R_i^2 \lambda^2 B(R_i) \tag{9}$$

and so, the fluxoid Φ is given as

$$2\pi R_i \lambda\, m v_s(R_i) + \frac{e}{c}\pi R_i^2 \lambda^2 B(R_i) = \frac{e}{c}\Phi \tag{10}$$

From Maxwell's equation, we have

$$v_s(R_i) = R_i \lambda \omega - \lambda \frac{e}{mc} B'(R_i) \tag{11}$$

and so,

$$-b'(R_i) + \frac{R_i}{2} b(R_i) = \frac{\Phi}{2\pi \lambda^2 R_i} \tag{12}$$

Now, since

$$-I_0' + \frac{r}{2} I_0 = \frac{-r}{2} I_2 \tag{13}$$

and since analogous expressions hold for K, we have, finally,

$$b(r) = \frac{\Phi}{\pi R_i^2 \lambda^2} \frac{I_0(R_0)\,K_0(r) - K_0(R_0)\,I_0(r)}{D_{20}} + b(R_0)\frac{I_2(R_i)\,K_0(r) - K_2(R_i)\,I_0(r)}{D_{20}} \tag{14}$$

where

$$D_{20} = I_2(R_i)\,K_0(R_0) - I_0(R_0)\,K_2(R_i)$$

and λR_0 is the outer radius of the cylinder.

It is now important to notice that the first term in Eq. (14) corresponds to a "frozen in" part of $b(r)$ as a result of fluxoid conservation. This term is unaffected by either the rotational state of the superconductor or the external field present outside the superconductor, these contributing only through the term $b(R_0)$. We point out again that Φ is fixed at the time the cylinder becomes superconducting and is unaffected by whatever happens to the cylinder after that time, so long as it remains superconducting.

IV. DETERMINATION OF THE FLUXOID

The field in the hollow, $B(R_i)$, for a cylinder of arbitrary dimensions, depends on all three of the relevant physical quantities: the external field, the velocity of rotation, and the fluxoid. If, however, the superconductor is stationary and there is no applied field, then $B(R_i)$ depends only on the fluxoid and is

$$\frac{\Phi}{\pi(R_i\lambda)^2}\left(\frac{1}{1-\dfrac{2}{i\,R_i}\dfrac{D_{10}}{D_{00}}}\right) \tag{15a}$$

here,

$$D_{mn} = I_m(R_i)\,K_n(R_0) - K_m(R_i)\,I_n(R_0) \tag{15b}$$

[If we now assume that $R_i \gg 1$ (diameter much greater than a penetration depth) and $R_0 - R_i = \delta \ll 1$ (thickness much less than a penetration depth), then, to within first order in δ, we find

$$\begin{aligned} D_{00} &= \frac{-2\delta}{i\,\pi\,R_i} \\ D_{10} &= \frac{2}{\pi}\frac{1}{R_i} \end{aligned} \tag{16}$$

The flux $\mathcal{F}$ associated with the fluxoid Φ, then, has the form

$$\mathcal{F} = B(R_i)\left[\pi(R_i\lambda)^2\right] \tag{17a}$$

$$= \Phi\left(\frac{1}{1+\dfrac{2}{R_i\delta}}\right) \tag{17b}$$

and it agrees with expressions previously obtained by other methods (Ref. 4).* However, as we see, this expression is already a consequence of London's equations, and to obtain it, it is not necessary to invoke the Ginzburg–Landau equations as might be inferred from Ref. 4.]

However, in the limit $\delta \gg 1$ (thickness much greater than penetration depth), the fluxoid ceases to depend on the external field. In this limit, $b(r)$ takes on a simple form:

$$\sinh(R_i - R_0)\,b(r) = -\frac{\Phi}{\pi R_i^2\,\lambda^2}\sqrt{\frac{R_i}{r}}\sinh(R_0 - r) + b(R_0)\sqrt{\frac{R_0}{r}}\sinh(R_i - r) \tag{18}$$

Thus, in this approximation,*

$$b(R_i) = +\frac{\Phi}{\pi R_i^2\,\lambda^2} \tag{19a}$$

and

$$B(R_i) = -\frac{2mc}{e}\,\omega + \frac{\Phi}{\pi(R_i\lambda)^2} \tag{19b}$$

We now notice from Eq. (19b) that if the superconductor with no persistent current ($\Phi = 0$) is rotated, then the field becomes $-B_0$ in the hollow of the cylinder; this is precisely what Hildebrandt (Ref. 1) found. However, suppose now that the cylinder is made superconducting while it is rotating. What value of the fluxoid is then established? Hildebrandt finds $B(R_i) = 0$, so that according to Eq. (19b),

$$B_0 = \frac{\Phi}{\pi R_i^2\,\lambda^2} \tag{20a}$$

This means that

$$\Phi = \pi R_i^2\,\lambda^2\,B_0 \tag{20b}$$

so that when the superconductor is stopped,

$$b(R_i) = B(R_i) = B_0 \tag{20c}$$

The following argument demonstrates how this value of the fluxoid is generated. The argument is based on the way the cylinder becomes superconducting. If we suppose that in the cylinder there are nucleation sites where the material first becomes superconducting (Ref. 5), we can imagine little spheres, say, of superconducting material, that grow with time and finally coalesce to make the entire material superconducting.

*Mercereau and Hunt also realize that London's theory suffices to obtain the relation that holds between flux and fluxoid in a thin cylinder.

*From Eq. (18), we can also see that if the cylinder wall is thick enough compared to the penetration distance, $b(r) = 0$ in the wall within a certain distance from either edge. In this region, then, $B(r) = -B_0$ regardless of the values of either the fluxoid or the external field, and again we recover the Meissner effect for a rotating superconductor.

Since they have an angular velocity, these spheres will have the proper surface current to generate the field $-B_0$, as soon as their diameter exceeds the penetration depth. If an applied field is present, an additional surface current will appear in order to maintain the internal field at the value $-B_0$. If the surface currents are added up as the spheres coalesce, currents will be found to cancel except on the *inside* and *outside surfaces* of the cylinder. On these surfaces, the currents will be equal and opposite, and, as a result, the applied field will appear in the hollow of the superconductor.

The argument that this field configuration is thermodynamically stable is based on an expression introduced for the free energy of a rotating superconductor in an applied field. We write the free energy as a sum of two terms. The first term, $g_s(T)V$, is the free energy of the stationary superconductor in zero field; it is simply the temperature (T)-dependent free-energy density multiplied by the volume V of the superconductor. To this, we add a second term which, for thick-walled superconductors, is simply the magnetic energy generated by rotation of the superconductor. This energy represents the isothermal work done by the superconductor on the magnetic field. Further discussion of this expression will be presented in Ref. 3. Here, we limit ourselves to applying it to the hollow cylinder to determine the value of the fluxoid that leads to the minimum magnetic energy generated by the cylinder when it is made to rotate.

The argument is essentially this: Outside the cylinder, there is the applied field B_∞; inside the wall, if it is thick enough, the field is $-B_0$ and is unaffected by either the field in the hollow of the cylinder or the field applied. Clearly, then, for a thick-walled cylindrical superconductor with angular velocity ω_0, the magnetic energy generated is a minimum if $B(R_i) - B_\infty = 0$ in its hollow. From Eq. (19b), it is clear that

$$\Phi = +\pi B_0' \, (R_i\lambda)^2$$

$$B_0' = \frac{2mc}{e}\,\omega_0' \tag{21}$$

$$\omega_0' = \omega_0 + \frac{e}{2mc}\,B_\infty$$

It might be of some interest to calculate the proper value of the fluxoid for the thin-walled cylinder. However, we forego that calculation here and regard Eqs. (21) as our main result.

We now see that the field $B(R_i)$ in the rotating cylinder depends on its "history" through the term $(2mc/e)\,\omega_0'$, and so the field is not a unique function of the angular velocity of the cylinder. This is at least in qualitative accord with Hildebrandt's experiments (Ref. 1).

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